# A RIGIDITY CONDITION FOR GENERATORS IN STRONGLY CONVEX DOMAINS

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ABSTRACT. Let $F$ be an infinitesimal generator of a semigroup of holomorphic self-maps in a smooth strongly convex subdomain $D$ of $\mathbb{C}^n$. We prove that $F \equiv 0$ on $D$ if $F$ vanishes in angualr sense at a boundary point up to third order.

A *semigroup* $\Phi_t$ of holomorphic maps in a subdomain $D$ of $\mathbb{C}^n$ is a continuous homomorphism from the additive semigroup $(\mathbb{R}^+, +)$ into the semigroup $Hol(D,D)$ of holomorphic self-maps of $D$, respect to the operation of composition, endowed with the topology of uniform convergence on compacts subsets. We know ( [2, Section 2.5.3], [24] and [22]) that the function $[0, +\infty) \ni t \mapsto \Phi_t \in Hol(D,D)$ is analytic, and that to each such semigroup there corresponds a vector field $F : D \to \mathbb{C}^n$ (as usual we identify $\mathbb{C}^n$ with its tangent space), such that $\frac{\partial \Phi_t}{\partial t} = F(\Phi_t)$ (It should be noted that the book [24] uses a different sign convention, so some formulas may appear a bit different). This vector field is usually called the *infinitesimal generator* of the semigroup. It is a *semicomplete* vector field, in the sense that each maximal solution $\gamma_z$, with $\gamma(0) = z$, can be extended up to $+\infty$. On the other hand, let $D$ be a subdomain of $\mathbb{C}^n$ and $F : D \to \mathbb{C}^n$ a holomorphic map; if, for each $z \in D$, the Cauchy problem given by

$$\begin{cases} \dfrac{\partial \Phi_z(t)}{\partial t} = F(\Phi_z(t)) \\ \Phi_z(0) = z \end{cases}$$

has a unique maximal solution defined on $[0, +\infty)$, then $F$ is the infinitesimal generator of a unique one-parameter semigroup of holomorphic self-maps of $D$.

Not all holomorphic maps defined on a domain $D$ are infinitesimal generators of some semigroups of holomorphic self-maps of $D$; for $D = \Delta$ (the open unit disc of $\mathbb{C}$), we have the following powerful representation formula found by Berkson and Porta ( [7] and the books [2], [24] and [22]):

**Theorem 0.1** (Berkson-Porta). *Let $f : \Delta \to \mathbb{C}$ be a holomorphic function. Then $f$ is the infinitesimal generator of a semigroup of holomorphic self-maps of $\Delta$ if and only if there exists $b \in \overline{\Delta}$ and a holomorphic function $p : \Delta \to \mathbb{C}$, such that $\Re p \geq 0$ and*

$$(0.1) \qquad\qquad f(\xi) = (\xi - b)(\overline{b}\xi - 1)p(\xi).$$

*Furthermore $b$ and $p : \Delta \to \mathbb{C}$ are uniquely determined by $f$.*

For us a *rigidity condition* is a sufficient condition on $F$ ensuring $F$ has to vanish on $D$. Since the vanishing of $F$ on $D$ is equivalent to the semigroup generated by $F$ being trivially the identity map on $D$, a rigidity condition on $F$ may be linked to some condition on a self map of $D$ that forces it to be the identity map. In this line of arguments D. M. Burns and S. G. Krantz published an article ( [12]) providing a condition under which a holomorphic map $\phi : \Delta \to \Delta$ is the identity:

**Theorem 0.2.** *Let $\phi : \Delta \to \Delta$ be a holomorphic map and $\tau \in \partial\Delta$. If*
$$\phi(\xi) = \tau + (\xi - \tau) + O(|\xi - \tau|^4)$$
*for $\xi \to 1$, then $\phi \equiv id_\Delta$.*

Furthermore they extend this result to $\mathbb{B}^n$, the open unit ball of $\mathbb{C}^n$ and even to more general domains; this theorem is a kind of Schwarz lemma at the boundary.

In recent years R. Tauraso and F. Vlacci ( [25]) and F. Bracci, R. Tauraso and F. Vlacci ( [11]) improved this result showing that the unrestricted limit can be replaced by an angular one or, being equivalent in $\Delta$, by the limit along a non tangential curve. They used this to prove, in the last paper cited above, an identity principle for commuting self-maps of $\Delta$. Further generalizations of these results to more general situations can be found for instance in [14], [6] and [21].

As first discovered by A. Korányi and E. M. Stein ( [17] and [18]), the right (for function theory) generalization to the unit ball in several complex variables of angular regions in $\Delta$ is not a conic region, but the following set:
$$K(\tau, M) = \{z \in \mathbb{B}^n \mid \frac{|1 - (z, \tau)|}{1 - \|z\|} M\},$$
where $\tau$ is a point in the boundary of $\mathbb{B}^n$, $(\cdot, \cdot)$ is the standard hermitian product in $\mathbb{C}^n$ and $M1$. These regions are called *Korányi regions of center $\tau$ and amplitude $M$*. In $\Delta$ these regions are just egg-shape sectors with a corner at $\tau$, symmetric with respect to the line segment from 0 to $\tau$. In $\mathbb{B}^n$ they are angular only along the direction normal to the boundary of $\mathbb{B}^n$ and tangential in all others directions. Korányi regions can be introduced also in Hilbert spaces. Using Korányi regions we can define *K-limit* at any point $\tau$ in the boundary of $D$; we say that $F : D \to \mathbb{C}^n$ has $K$-limit $L$ and we write
$$K\text{-}\lim_{z \to \tau} F(z) = L$$
if $\lim F(z) = L$ for $z \to \tau$ inside the Korányi region $K(\tau, M)$ with center $\tau$, for any $M1$. In $\Delta$ for a function to have $K$-limit, is the same that to have angular limit.

In 2007 M. Elin, M. Levenshtein, S. Reich, D. Shoikhet ( [13]) proved the following couple of theorems:

**Theorem 0.3.** *Let $f : \Delta \to \mathbb{C}$ be the infinitesimal generator of a semigroup of holomorphic self-maps of $\Delta$. If*

(0.2) $$K\text{-}\lim_{\xi \to \tau} \frac{f(\xi)}{|\xi - 1|^3} = 0,$$

*then $f \equiv 0$ on $\Delta$.*

**Theorem 0.4.** *Let $\mathbb{B}$ be the unit ball in a Hilbert space $\mathbf{H}$, $\tau \in \partial\mathbb{B}$ and $F : \mathbb{B} \to \mathbf{H}$ the infinitesimal generator of a semigroup of holomorphic self-maps of $\mathbb{B}$. If*

(0.3) $$K\text{-}\lim_{z \to \tau} \frac{F(z)}{\|z - \tau\|^3} = 0,$$

*then $F \equiv 0$.*

They prove the former using dynamics properties of $f$ and unicity of $b$ and $p : \Delta \to \mathbb{C}$ in (0.1). Their proof of the latter uses automorphisms of the unit ball in a Hilbert space to reduce to the one dimensional case. So, on one side it is clear that a weaker limit than the unrestricted one is sufficient to guarantee a rigidity condition, on the other the intervention of Kobayashi distance in the ball suggests possible generalizations involving the Kobayashi distance in more general domains.

In [3], Abate found a generalization of Korányi region to any smooth strictly convex domain $D$ of $\mathbb{C}^n$. That is, given $\tau$ in the boundary of $D$ and $x$ in $D$, a *Korányi region of center $\tau$, pole $x$ and amplitude $M$* is the set of points

$$K^D(\tau, p, M) = \{z \in D \mid \lim_{w \to \tau}[k_D(z,w) - k_D(p,w)] + k_D(p,z)\log M\}.$$

Here $k_D$ is the Kobayashi distance in $D$. In [1] Abate proved that the previous limit exists and in [3] he proved that $K^{\mathbb{B}^n}(\tau, 0, M)$ and $K(\tau, M)$ are the same set (see also [2, Chapter 2.7] and [5]). We now recall the notions needed to understand the statement of Theorem 0.5 and refer to section 1 for more details and bibliography.

Again, $D$ denotes a smooth domain of $\mathbb{C}^n$ and $\tau$ a point in the $\partial D$, the boundary of $D$. We call a continuous curve $\alpha : [0,1) \to D$ a *$\tau$-curve* if $\lim_{t \to 1^-} \alpha(t) = \tau$. A *conic region* at $\tau$ of amplitude $c$ is the set of points

$$\{z \in D \mid \|z - \tau\| \le c\,\mathrm{dist}(z, \partial D)\}$$

for some $c1$ where $\mathrm{dist}(\cdot, \cdot)$ denotes the euclidean distance in $\mathbb{C}^n$. A *non-tangential $\tau$-curve* is a $\tau$-curve that lies inside some conic region at $\tau$. Given a function $F : D \to \mathbb{C}^n$, we say that $F$ has *non-tangential limit $L$* at $\tau$ and we write

$$\angle \lim_{z \to \tau} F(z) = L$$

if $\lim_{t \to 1^-} F(\alpha(t)) = L$ along any non-tangential $\tau$-curve. For comprehensive general reference on non-tangential limits we refer to [2], [3], [5] and [23].

The main theorem of this paper is the following.

**Theorem 0.5.** *Let $D$ be a bounded smooth strongly convex domain of $\mathbb{C}^n$; $F : D \to \mathbb{C}^n$ be the infinitesimal generator of a semigroup of holomorphic self-maps of $D$ and $\tau \in \partial D$. If*

$$\angle \lim_{z \to \tau} \frac{F(z)}{\|z - \tau\|^3} = 0,$$

*then $F \equiv 0$ in $D$.*

Theorem 0.5 is stronger than Theorem 0.4 even in the ball, since, as we shall explain better in the next section, $K$-limits are stronger than non-tangential ones, since each non-tangential $\tau$-curve is eventually contained in some Korányi region at $\tau$. We furtermore remark that the last theorem holds also if $D$ has boundary of class $\mathcal{C}^k$, $k \ge 14$ and in bounded strictly linearly convex domain with the same regularity at the boundary.

This paper is organized as follows. In section 1 we shall recall some standard tools needed for the proof, mainly the Lempert theory of complex geodesics. In section 2 we will prove, following [8] some results allowing us to use Theorem 0.4 about the infinitesimal generators of continuous semigroups of holomorphic self-maps of the unit disc $\Delta \subseteq \mathbb{C}$. In section 3 we shall prove our main result.

Finally I would like to thank mainly Marco Abate, my research director, for guiding me in studying this topic and problem, and Filippo Bracci, Simeon Reich for their suggestions.

## 1. General Framework

Let $D$ be a bounded, smooth, strongly convex domain of $\mathbb{C}^n$ and denote the Kobayashi distance in $D$ by $k_D$. A *complex geodesic* is a holomorphic map $\varphi : \Delta \to D$ which is an isometry beetween $k_\Delta = \omega$, the Poincaré distance in $\Delta$, and $k_D$. Any complex geodesic extends smoothly to the boundary, maps $\partial \Delta$ in $\partial D$ and further for each $\xi$ in $\partial \Delta$, $\varphi'(\xi)$ is transversal to the boundary of $D$.

To each complex geodesic $\varphi$ is associated a "*dual map*" (see [2, Chapter 2.6]), $\tilde{\varphi} : \overline{\Delta} \to \mathbb{C}^n$, holomorphic in $\Delta$, smooth up to the boundary, such that $\tilde{\varphi}(\xi) = \xi h(\xi) \partial r(\varphi(\xi))$ where $\xi$ is in $\partial \Delta$ and $h : \partial \Delta \to \mathbb{R}$ is a smooth and positive function. The map $\tilde{\varphi}$ is determined up to a positive constant which we normalize by requiring that $\langle \varphi', \tilde{\varphi} \rangle \equiv 1$ where $\langle \cdot, \cdot \rangle$ denotes the standard bilinear (not Hermitian!) form in $\mathbb{C}^n$.

Using $\tilde{\varphi}$ we can define a left inverse for $\varphi$: the holomorphic map such that for each $z$ in $D$, there is only one $\xi \in \Delta$ which solves the equation $\langle z - \varphi(\xi), \tilde{\varphi}(\xi) \rangle = 0$. We call $\tilde{\rho}_\varphi : D \to \Delta$ the map obtained in this way. It can be proved that $\tilde{\rho}_\varphi \circ \varphi \equiv id$ in $\Delta$. The map $\tilde{\rho}_\varphi$ even extends smoothly up to the boundary of $D$ and the fibers of $\tilde{\rho}_\varphi$ are intersections of $D$ with complex hyperplanes .

We put $\rho_\varphi = \varphi \circ \tilde{\rho}_\varphi$; then $\rho_\varphi \circ \rho_\varphi = \rho_\varphi$ and $\rho_\varphi \circ \iota_{\varphi(\Delta)} = id_{\varphi(\Delta)}$. The map $\rho_\varphi$ extends smoothly up to the boundary and $\tilde{\rho}_\varphi(\overline{D} \setminus \varphi(\partial \Delta)) \subseteq \Delta$. From the definition of $\varphi$, being it injective, it follows that the fibers of $\rho_\varphi$ are also intersections with $D$ of complex affine hyperplanes. We call $(\varphi, \rho_\varphi, \tilde{\rho}_\varphi)$ the *projection device* associated to the complex geodesic $\varphi$.

One of the consequences of Lempert's work is that the Kobayashi distance $k_D$ is smooth outside the diagonal of $D \times D$. Furthermore for any couple of points $z$, $w$ in $\overline{D}$ there exists just one complex geodesic $\varphi$, modulo precomposition by automorphisms of $\Delta$, such that $\{z, w\} \subseteq \varphi(\overline{\Delta})$. For complete statements and proofs we refer to [2, Chapter 2.6], [16, Chapter 5] and [15, Chapter 4] and, to the original works [19], [20].

In studying global iteration theory in convex domains, M. Abate introduced the concept of horosphere in the frame of invariant distances and this notion proves to be very useful. Let $D$ be as above, $\tau$ a point in $\partial D$ and $p$ a point in $D$. The horosphere of center $\tau$, pole $p \in D$ and radius $R$ is defined as the set

$$E(\tau, p, R) = \{z \in D \mid \lim_{x \to \tau} [k_D(z, x) - k_D(p, x)] \frac{1}{2} \log R\}.$$

It was proved in [1] that the limit exists, and horospheres are convex subsets of $D$ (see also [2, Theorem 2.6.47]). In the unit disc of $\mathbb{C}$ they are discs internally tangents to the boundary of the unit disc in the center of the horosphere. In the unit ball of $\mathbb{C}^n$ they are ellipsoids internally tangents to the boundary of the unit ball in the center of the horosphere. Recently it has been showed ( [26], [10, Section 4] and [9, Remark 6.4]) that horospheres in bounded smooth strongly convex domains are smooth and strongly convex (and thus strongly pseudoconvex), except at the center and have a global defining function ( [9, Theorem 6.3]). Precisely they are sublevel sets of the pluricomplex Poisson kernel in $D$ (see [10]).

The last tool we need is restricted limit. In many problems where the central feature is the boundary behavior of holomorphic maps, they prove to be very useful. We recall the basic definitions and proprieties, previously stated in the introduction. Let $D$, $\tau$ and $p$ be as above. As we said *conic region at* $\tau$ is the set of points

$$\{z \in D \mid \|z - \tau\| \leq c \, \mathrm{dist}(z, \partial D)\}$$

for some $c1$, where $\mathrm{dist}(\cdot, \cdot)$ denotes the euclidean distance in $\mathbb{C}^n$. A *Korányi region* centered at $\tau$, with pole $p$ and amplitude $M1$ is the set

$$K(\tau, p, M) = \{z \in D \mid \lim_{x \to \tau}[k_D(z, x) - k_D(p, x)] + k_D(z, p) \log M\}.$$

Recall again that a $\tau$-*curve* is a map $\alpha : [0, 1) \to D$ such that $\lim_{t \to 1^-} \alpha(t) = \tau$. A *non-tangential* $\tau$-curve is a $\tau$-curve that lies inside some conic region at $\tau$. The prototype of a non-tangential $\tau$-curve is $\varphi(r)$ for $r$ in $[0, 1)$, where $\varphi$ is a complex geodesics such that $\varphi(1) = \tau$.

A function $F : D \to \mathbb{C}^n$ has *non-tangential limit* $L$ at $\tau$ and we denote it by

$$\angle \lim_{z \to \tau} F(z) = L$$

if $\lim_{t \to 1^-} F(\alpha(t))$ exists and is the same for any non-tangential $\tau$-curve.

We say that $F : D \to \mathbb{C}^\ltimes$ has $K\text{-}\lim L$ and we write

$$K\text{-}\lim_{z \to \tau} F(z) = L$$

if $\lim F(z) = L$ for $z \to \tau$ whitin $K(\tau, M)$, for any $M1$.

The relation between these limits is the following: if $K\text{-}\lim_{z \to \tau} F(z) = L$ then $\angle \lim_{z \to \tau} F(z) = L$. This implication is generally strict.

For all the latter definitions and statements we refer the reader to [2, Chapter 2.7], [3] and [5] for the general case and to [23] for the unit ball.

## 2. Infinitesimal Generator

We start this section with a definition that will simplify the following statements.

**Definition 2.1.** Let

$$HolG(D) = \{F \in Hol(D, \mathbb{C}^n) \mid \text{F is the infinitesimal generator of a one}$$

$$\text{parameter semigroup of holomorphic self-maps of D}\}.$$

A recent paper ( [8]) provides us with the following useful condition under which a map $F \in Hol(D, \mathbb{C}^n)$ is in $HolG(D)$, namely:

**Theorem 2.2.** [8, Theorem 3.5] *Let $D$ be a smooth strongly convex domain of $\mathbb{C}^n$, $F \in Hol(D, \mathbb{C}^n)$; then $F \in HolG(D)$ if and only if $\forall z, w \in D$, $z \neq w$, we have*

$$(dk_D)_{(z,w)}(F(z), F(w)) \leq 0.$$

This property of $F$ says more or less that the flow generated goes inside each ball in the Kobayashi distance. The proof uses a few very basic notions of potential theory but can quite easily be adapted to avoid them.

A similar condition with the Kobayashi metric in place of the Kobayashi distance was found in [4]. Both of these are related to the notion of $k_D$-*monotonicity* as introduced, for instance, in [22].

The following couple of results are implicit in the proof of [8, Proposition 4.5]. In order to make this paper self contained we shall state and prove them here without using potential theoretic notions.

**Lemma 2.3.** *Let $z \in D$, let $B_{k_D}(z, R)$ be a Kobayashi-ball of $D$ centered in $z$ and with radius $R$, and $w \in \partial B_{k_D}(z, R)$; furthermore let $(\rho_\varphi, \tilde{\rho}_\varphi, \varphi)$ be a projection device such that $z, w \in \varphi(\Delta)$. Then*

$$T_w^{\mathbb{C}} \partial B_{k_D}(z, R) = \ker(d\rho_\varphi)_w.$$

*Proof.* Observe that $d\rho_\varphi = \partial \rho_\varphi$ because $\rho_\varphi$ is holomorphic. The fibers of $\rho_\varphi$ are intersection with $D$ of complex affine hyperplanes and we have $\ker(d\rho_\varphi)_w = \ker(\partial \rho_\varphi)_w = \{y - w \mid y \in \rho_\varphi^{-1}(w)\}$. Now $\rho_\varphi(z) = z$, $\rho_\varphi(w) = w$ because $\rho_\varphi$ is a retraction on $\varphi(\Delta)$; then $\rho_\varphi(B_{k_D}(z, R)) \subseteq B_{k_D}(z, R)$ because $\rho_\varphi$ is holomorphic and contracts the Kobayashi distance. So we have $\rho_\varphi(B_{k_D}(z, R)) \subseteq B_{k_D}(z, R) \cap \varphi(\Delta)$, thus no point $y \in \rho_\varphi^{-1}(w)$ can lie inside $B_{k_D}(z, R)$. We know that the balls in the Kobayashi distance in $D$ are convex subsets with smooth boundary, so each point in the boundary has a unique real tangent hyperplane and also a unique complex tangent hyperplane. Then the complex hyperplane $\{y \in \mathbb{C}^n \mid \rho_\varphi(y) = w\}$ is the complex tangent hyperplane at $\partial B_{k_D}(z, R)$ in $w$ and thus

$$T_w^{\mathbb{C}} \partial B_{k_D}(z, R) = \ker(d\rho_\varphi)_w.$$

$\square$

**Lemma 2.4.** *For any complex geodesic $\varphi : \Delta \to D$, with associated $(\rho_\varphi, \tilde{\rho}_\varphi, \varphi)$ projection device and for any $z = \varphi(\xi) \in \varphi(\Delta)$, $w = \varphi(\eta) \in \varphi(\Delta)$, with $z \neq w$ and vectors $u \in T_z^{\mathbb{C}}(D)$, $v \in T_w^{\mathbb{C}}(D)$ we have*

$$(dk_D)_{(z,w)}(u, v) = (dk_\Delta)_{(\xi,\eta)}((d\tilde{\rho}_\varphi)_z(u), (d\tilde{\rho}_\varphi)_w(v)).$$

*Proof.* First of all, from the fact that $\varphi$ is an isometry, we have, $k_D(\varphi(\delta), \varphi(\theta)) = k_\Delta(\delta, \theta)$, for all $\delta, \theta \in \Delta$, and hence

$$
\begin{aligned}
d(k_D)_{(\varphi(\xi),\varphi(\eta))}[(d\varphi)_\xi(\zeta), (d\varphi)_\eta(\sigma)] &= d_x(k_D(x, y))_{(\varphi(\xi),\varphi(\eta))}[(d\varphi)_\xi(\zeta), 0] \\
&\quad + d_y(k_D(x, y))_{(\varphi(\xi),\varphi(\eta))}[0, (d\varphi)_\eta(\sigma)] \\
&= (dk_\Delta)_{(\xi,\eta)}(\zeta, \sigma)
\end{aligned}
$$

for any $\zeta$ in $T_\xi(\mathbb{C}) \cong \mathbb{C}$ and $\sigma$ in $T_\eta(\mathbb{C}) \cong \mathbb{C}$. We claim that

$$(2.1) \qquad d_x(k_D(x, y))_{(z,w)}(u, 0) = d_x(k_D(x, y))_{(z,w)}[(d\rho_\varphi)_z(u), 0]$$

and

$$(2.2) \qquad d_y(k_D(x, y))_{(z,w)}(0, v) = d_y(k_D(x, y))_{(z,w)}[0, (d\rho_\varphi)_w(v)],$$

for any $(u, v)$ in $T_z^{\mathbb{C}}(D) \times T_w^{\mathbb{C}}(D)$. We shall prove only (2.2), the other proof is similar. Since $\rho_\varphi$ is a holomorphic retraction on $\varphi(\Delta)$, we have $d\rho_\varphi = \partial \rho_\varphi$ and $(d\rho_\varphi)_w \circ (d\rho_\varphi)_w = (d\rho_\varphi)_w$; thus $d\rho_\varphi$ is a linear projection in $T_w^{\mathbb{C}}(D)$ and we have a holomorphic splitting

$$T_w^{\mathbb{C}}(D) = d\rho_\varphi(T_w^{\mathbb{C}}(D)) \oplus \ker(d\rho_\varphi)_w.$$

Using the previous lemma we have

$$(2.3) \qquad \ker(d\rho_\varphi)_w = T_w^{\mathbb{C}} \partial B_{k_D}(z, R) = \ker d_y(k_D(x, y))_{(z,w)}$$

where $R = k_D(z, w)$. This concludes the proof of the claim. Hence we have

$$
\begin{aligned}
(dk_D)_{(z,w)}(u, v) &= d_x(k_D(x, y))_{(z,w)}(u, 0) \\
&\quad + d_y(k_D(x, y))_{(z,w)}(0, v) \\
&= d_x(k_D(x, y))_{(z,w)}[(d\rho_\varphi)_z(u), 0] \\
&\quad + d_y(k_D(x, y))_{(z,w)}[0, (d\rho_\varphi)_w(v)] \\
&= d_x(k_D(x, y))_{(z,w)}d\varphi_\xi[(d\tilde\rho_\varphi)_z(u),] \\
&\quad + d_y(k_D(x, y))_{(z,w)}[0, d\varphi_\eta(d\tilde\rho_\varphi)_w(v)] \\
&= (dk_\Delta)_{(\xi,\eta)}((d\tilde\rho_\varphi)_z(u), (d\tilde\rho_\varphi)_w(v)).
\end{aligned}
$$

$\square$

We are ready for the main theorem of this section: it provides us with a link between an infinitesimal generator in $D$ and its projections in the images of complex geodesics. It was proved in [8, Proposition 4.5], but here we give a proof independent of potential theoretic notions, along the same lines.

**Theorem 2.5.** *Let $D$ be a bounded, smooth, strongly convex domain in $\mathbb{C}^n$ and $F : D \to \mathbb{C}^n$ a holomorphic map. Then $F \in HolG(D)$ if and only if for any complex geodesics $\varphi : \Delta \to D$ the vector field on $\Delta$ given by $(d\tilde\rho_\varphi)_\xi(F(\varphi(\xi)))$ is in $HolG(\Delta)$.*

*Proof.* Consider a complex geodesic $\varphi$ and set $f_\varphi(\xi) := (d\tilde\rho_\varphi)_{\varphi(\xi)}(F(\varphi(\xi)))$. So we have to prove that $F \in HolG(D)$ if and only if $f_\varphi \in HolG(\Delta)$ for all complex geodesics $\varphi$. From the previous lemma it follows that

$$
\begin{aligned}
(dk_D)_{(\varphi(\xi),\varphi(\eta))}&(F(\varphi(\xi)), F(\varphi(\eta))) \\
&= (dk_\Delta)_{(\xi,\eta)}((d\tilde\rho_\varphi)_{\varphi(\xi)}(F(\varphi(\xi))), (d\tilde\rho_\varphi)_{\varphi(\eta)}(F(\varphi(\eta)))) \\
&= (dk_\Delta)_{(\xi,\eta)}(f_\varphi(\xi), f_\varphi(\eta)),
\end{aligned}
$$

for any $\xi, \eta$ in $\Delta$. Now on one hand, if $\varphi$ is any complex geodesics and $F \in HolG(D)$ then $f_\varphi \in HolG(\Delta)$ by by Theorem 2.2. On the other hand for any $z, w$ in $D$ exists a unique complex geodesics $\varphi : \Delta \to D$ such that $\varphi(\xi) = z$ and $\varphi(\eta) = w$ for some $\xi, \eta \in \Delta$. Thus we have the desired conclusion $\square$

## 3. Rigidity Condition

In this section we are going to prove the main theorem of our paper. Before going further we need a preparatory lemma. In [9, Theorem 6.3] it is proved that horospheres of radius $R$ and fixed center $p$ are $R$-sublevel set of the global smooth function $\Omega_{\tau,p}$, which is the pluricomplex Poisson kernel defined on $D$. Here we do not need any particular property of $\Omega_{\tau,p}$ other than its smoothness and strict pseudo-convexity of its sublevel sets, and the following

**Lemma 3.1.** *Let $\tau \in \partial D$, $p \in D$ and let $\Omega_{\tau,p} : D \to (0, +\infty)$ be the smooth convex function, defined above, whose sublevel sets are $\{x \in D \mid \Omega_{\tau,p}(x)r\} = E(\tau, p, R)$, the horospheres of center $\tau$ and pole $p$, for $r0$. Furthermore let $z$ be any point in $\partial E(\tau, p, R)$ and let $\varphi$ be a complex geodesic such that $z, \tau \in \overline{\varphi(\Delta)}$ and let $(\varphi, \rho_\varphi, \tilde\rho_\varphi)$ be its projection device. Then*

$$
\tag{3.1} \ker(d\rho_\varphi)_z = T_z^{\mathbb{C}}\partial E(\tau, p, R) = \ker(\partial\Omega_{\tau,p})_z
$$

*where $R = \Omega_{\tau,p}(z)$.*

*Proof.* We have

$$\lim_{y \to \tau} [k_D(\rho_\varphi(w), y) - k_D(p, y)] = \lim_{\varphi(\Delta) \ni y \to \tau} [k_D(\rho_\varphi(w), y) - k_D(p, y)]$$
$$\leq \lim_{\varphi(\Delta) \ni y \to \tau} [k_D(w, y) - k_D(p, y)]$$
$$= \lim_{y \to \tau} [k_D(w, y) - k_D(p, y)],$$

for any $w$ in $D$. Thus observing that $\rho_\varphi \circ \iota_{\varphi(\Delta)} = id_{\varphi(\Delta)}$, we have $\rho_\varphi(E(\tau, p, R)) = E(\tau, p, R) \cap \varphi(\Delta)$ for every $R0$. Hence no point of the fiber $\{w \in D \mid \rho_\varphi(w) = z\}$ can lie inside $E(\tau, p, R)$. Since $\rho_\varphi$ is holomorphic we have $d\rho_\varphi = \partial\rho_\varphi$. The fibers of $\rho_\varphi$ are intersections with $D$ of complex affine hyperplanes and we have

$$\ker(d\rho_\varphi)_z = \ker(\partial\rho_\varphi)_z = \{w - z \mid w \in \rho_\varphi^{-1}(z)\}.$$

As we remarked in Section 1 (see also [9], [10]), horospheres in a bounded smooth strongly convex domain $D$ are bounded smooth convex subdomain of $D$; so each point in the boundary of a horosphere has a unique real tangent hyperplane and also a unique complex tangent hyperplane. Hence the complex hyperplane $\{w \in \mathbb{C}^n \mid \rho_\varphi(w) = z\}$ is the complex tangent hyperplane at $\partial E(\tau, p, R)$ in $z$ and thus we can conclude that

$$\ker(d\rho_\varphi)_z = T_z^{\mathbb{C}} \partial E(\tau, p, R) = \ker(\partial\Omega_{\tau, p})_z.$$

$$\square$$

We can now state and prove the following;

**Theorem 3.2.** *Let $F \in HolG(D)$, $\tau \in \partial D$; furthermore for each projection device $(\varphi, \rho_\varphi, \tilde{\rho}_\varphi)$, let $f_\varphi(\xi) := (d\tilde{\rho}_\varphi)_{\varphi(\xi)}(F(\varphi(\xi)))$. Then $F \equiv 0$ in $D$ if and only if $f_\varphi \equiv 0$ in $\Delta$ for any complex geodesic $\varphi$ such that $\tau \in \varphi(\partial\Delta)$.*

*Proof.* The $(\Rightarrow)$ direction is trivial, so suppose that $f_\varphi \equiv 0$ in $\Delta$ for every complex geodesic $\varphi$ such that $\tau \in \varphi(\partial\Delta)$. Let $z \in D$ , $p \in D$; let $E(\tau, p, R)$ a horosphere such that $z \in \partial E(\tau, p, R)$ and $\varphi$ be a complex geodesics such that $z = \varphi(\xi)$ for some $\xi \in \Delta$ and $\tau \in \varphi(\partial\Delta)$. Thus we have

$$(d\rho_\varphi)_z(F(z)) = (d\varphi)_\xi(d\tilde{\rho}_\varphi)_{\varphi(\xi)}(F(\varphi(\xi)) = (d\varphi)_\xi f_\varphi(\xi) = 0$$

Thus from the identity (3.1) it follows that $(\partial\Omega_{\tau, p})_z F(z) = 0$. Deriving with respect to $\overline{\partial}$ we have $(\partial\overline{\partial}\Omega_{\tau, p})_z F(z) = 0$ because $\overline{\partial}_z F(z) = 0$ being $F$ holomorphic. Now from the previous Lemma 3.1, we know that $F(z) \in T_z^{\mathbb{C}} \partial E(\tau, p, R)$ where $R = \Omega_{\tau, p}(z)$. By [9, Remark 6.4] boundaries of horospheres are strictly pseudoconvex; so the Levi form $\partial\overline{\partial}\Omega_{\tau, p}$ restricted to their complex tangent spaces has to be positive definite. So we must have $F(z) = 0$. The arbitrariness of $z \in D$ implies $F \equiv 0$ in $D$ and this concludes the proof. $\square$

Now as an application of the previous result we can prove;

**Theorem 3.3.** *Let $D \subset\subset \mathbb{C}^n$ be smooth, strongly convex, $\tau \in \partial D$, $F \in HolG(D)$. If*

$$(3.2) \qquad \angle \lim_{z \to \tau} \frac{F(z)}{\|z - \tau\|^3} = 0.$$

*then $F \equiv 0$ in $D$.*

*Proof.* We use the same notation introduced in Theorem 3.2. Let $(\varphi, \rho_\varphi, \tilde{\rho}_\varphi)$ be a projction device at $\tau$ such that $\varphi(1) = \tau$. Let $\gamma(t) : [0, 1) \to \Delta$ any non-tangential 1-curve and $\beta(t) = \varphi(\gamma(t))$; clearly $\beta$ is a non tangential $\tau$-curve since $\varphi(\overline{\Delta})$ is transversal to the boundary of $D$. From the existence of the non-tangential limit in (3.2) and since $(d\tilde{\rho}_\varphi)_z$ depends continuously up to the boundary on $z \in D$, we have

$$\lim_{t \to 1} \frac{(d\tilde{\rho}_\varphi)_{\beta(t)} F(\beta(t))}{\|\beta(t) - \tau\|^3} = 0.$$

Since $f_\varphi(\xi) := d(\tilde{\rho}_\varphi)_{\varphi(\xi)}(F(\varphi(\xi)))$, we have $f_\varphi(t) := d(\tilde{\rho}_\varphi)_{\beta(t)}(F(\beta(t))$ and thus

$$\lim_{t \to 1} \frac{f_\varphi(\gamma(t))}{\|\beta(t) - \tau\|^3} = 0.$$

So we can conclude that

$$\lim_{t \to 1} \frac{f_\varphi(\gamma(t))}{|\gamma(t) - 1|^3} = \lim_{t \to 1} \frac{f_\varphi(\gamma(t))}{\|\beta(t) - \tau\|^3} \times \frac{\|\beta(t) - \tau\|^3}{|\gamma(t) - 1|^3} = 0$$

because the second ratio in the product remains bounded in a neighborhood of 1 being $\varphi$ $\mathcal{C}^\infty$ up to the boundary. Since $\gamma$ is any arbitrary non tangential 1-curve in $\Delta$, the previous limits implies also the angular limit in $\Delta$

$$\angle \lim_{\xi \to 1} \frac{f_\varphi(\xi)}{|\xi - 1|^3} = 0.$$

By Theorem 2.5 $f_\varphi$ is in $HolG(\Delta)$ and now using Theorem 0.3 (but see [13, Proposition 2] also) we have $f_\varphi \equiv 0$ in $\Delta$. The previous Theorem 3.2 and the arbitrariness of $\varphi$ allows us to conclude that $F \equiv 0$ on $D$. $\qquad\square$

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